

TORUS ACTIONS AND THE HALPERIN-CARLSSON CONJECTURE

YOSHINOBU KAMISHIMA AND MAYUMI NAKAYAMA

ABSTRACT. We give an affirmative answer to the Halperin-Carlsson conjecture for the homologically injective torus actions on closed manifolds. This class contains *holomorphic torus actions on compact Kähler manifolds*, *torus actions on compact Riemannian flat manifolds*.

1. INTRODUCTION

Recently the real Bott tower and its generalization have been studied by several people ([6], [13], [18], [15],[12]). A real Bott manifold is originally defined to be the set of real points in the Bott manifold [9]. Among several characterizations by group actions, the Halperin-Carlsson conjecture is true for real Bott manifolds. The Halperin-Carlsson torus conjecture says that if there is an almost free torus action T^k on a closed n -manifold M , the following inequality holds:

$$(1.1) \quad 2^k \leq \sum_{j=0}^n b_j.$$

Here $b_j = \text{rank } H_j(M; \mathbb{Z})$ is the j -th Betti number of M . See [19] for details and the references therein, see also [10].

Another characterization is that a real Bott manifold M is a euclidean space form (Riemannian flat manifold) admitting a torus action T^k with $k = \text{rank } H_1(M)$. It is conceivable whether the *Halperin-Carlsson conjecture* holds for compact euclidean space forms more generally.

By this motivation we revisit the classical results of the Calabi construction of euclidean space forms with nonzero b_1 [4] and the Conner-Raymond's injective torus actions [8]. In this direction, we shall introduce *injective-splitting action* of a torus T^k on closed manifolds more generally. Our purpose of this paper is to prove the Halperin-Carlsson conjecture for such torus actions affirmatively.

Let T^k be a k -dimensional torus ($k \geq 1$). Given an *effective* T^k -action on a closed manifold M , the orbit map at $x \in M$ is defined to be $\text{ev}(t) = tx$ ($\forall t \in T^k$). Put $\pi_1(T^k) = H_1(T^k; \mathbb{Z}) = \mathbb{Z}^k$ and $\pi_1(M) = \pi$. The map

Date: March 8, 2013.

2000 Mathematics Subject Classification. 53C55, 57S25, 51M10.

Key words and phrases. Torus action, Halperin-Carlsson Conjecture, Seifert fiber space, Riemannian flat manifold, Kähler manifold.

ev induces a homomorphism $\text{ev}_\# : \mathbb{Z}^k \rightarrow \pi$ and $\text{ev}_* : \mathbb{Z}^k \rightarrow H_1(M; \mathbb{Z})$ respectively. According to the definition of Conner-Raymond [8], if $\text{ev}_\#$ is *injective*, the action (T^k, M) is said to be *injective*. (Refer to [14, Theorem 2.4.2, also Subsection 11.1] for the definition to be independent of the choice of the base point $x \in M$.) Classically it is known that $\text{ev}_\#$ is injective for closed *aspherical* manifolds [7]. On the other hand, if $\text{ev}_* : \mathbb{Z}^k \rightarrow H_1(M; \mathbb{Z})$, the T^k -action is said to be *homologically injective* [8]. We have shown

Theorem A. *If T^k is a homologically injective action on a closed n -manifold M , then*

$$(1.2) \quad {}_k C_j \leq b_j.$$

In particular the Halperin-Carlsson conjecture is true.

To prove Theorem A we revisit the Conner and Raymond's work [8]. Then this is a consequence from *injective-splitting torus actions* more generally. See Theorem 2.3. We verify that the following actions are in fact injective-splitting torus actions.

Corollary B. *Every effective T^k -action on a compact n -dimensional euclidean space form M is homologically injective. Thus ${}_k C_j \leq b_j$, the Halperin-Carlsson conjecture (1.1) holds.*

We obtain a characterization of *holomorphic* torus actions originally observed by Carrell [5].

Corollary C. *Every holomorphic action of the complex torus $T_{\mathbb{C}}^k$ on a compact Kähler manifold is homologically injective. In particular, ${}_k C_j \leq b_j$, the Halperin-Carlsson conjecture holds.*

In Section 2, we introduce *injective-splitting actions* on closed manifolds and prove our main theorem 2.3. Using this theorem, we show Corollaries B and C. In Section 3 we shall give a proof concerning the existence of torus actions common to both the Calabi's theorem and the Conner-Raymond's theorem as our motivation (cf. Theorem 3.2).

Theorem E. *A compact n -dimensional euclidean space form M admits a homologically injective action of T^k with $k = \text{rank } H_1(M)$ in which $\text{rank } C(\pi) = \text{rank } H_1(M)$.*

2. INJECTIVE TORUS ACTIONS

Suppose (T^k, M) is an *injective action* on a closed manifold M . Let \tilde{M} be the universal covering space of M and denote $N_{\text{Diff}(\tilde{M})}(\pi)$ the normalizer of π in $\text{Diff}(\tilde{M})$. The conjugation homomorphism $\mu : N_{\text{Diff}(\tilde{M})}(\pi) \rightarrow \text{Aut}(\pi)$ defined by $\mu(\tilde{f})(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ ($\forall \gamma \in \pi$) induces a homomorphism φ which

has a commutative diagram:

$$\begin{array}{ccccc}
 & 1 & & 1 & & 1 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & C(\pi) & \longrightarrow & \pi & \xrightarrow{\mu} & \text{Inn}(\pi) \\
 & \downarrow & & \downarrow & & \downarrow \\
 (2.1) & C_{\text{Diff}(\tilde{M})}(\pi) & \longrightarrow & N_{\text{Diff}(\tilde{M})}(\pi) & \xrightarrow{\mu} & \text{Aut}(\pi) \\
 & \nu \downarrow & & \nu \downarrow & & \downarrow \\
 & \text{Diff}(M)^0 \leq \ker \varphi & \longrightarrow & \text{Diff}(M) & \xrightarrow{\varphi} & \text{Out}(\pi) \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 1 & & 1 & & 1
 \end{array}$$

(Compare [16].) Here $C_{\text{Diff}(\tilde{M})}(\pi)$ is the centralizer of π in $\text{Diff}(\tilde{M})$. As $T^k \leq \text{Diff}(M)^0$, we have a lift $\tilde{T}^k \leq C_{\text{Diff}(\tilde{M})}(\pi)$ to \tilde{M} . Note $\tilde{T}^k = \mathbb{R}^k$. For this, suppose some $S^1 \leq T^k$ lifts to S^1 (but not \mathbb{R}) on \tilde{M} . If $p : \tilde{M} \rightarrow M$ is the covering map which is equivariant; $p(tx) = t^m p(x)$ ($t \in S^1$) for some $m \in \mathbb{Z}$, chasing a commutative diagram

$$\begin{array}{ccc}
 \pi_1(S^1) & \xrightarrow{\text{ev}\#} & \pi_1(\tilde{M}) = 1 \\
 m \cdot \downarrow & & p\# \downarrow \\
 \pi_1(S^1) & \xrightarrow{\text{ev}\#} & \pi_1(M),
 \end{array}$$

it follows $\text{ev}\#(m\mathbb{Z}) = 1$. This contradicts the injectivity of $S^1 \leq T^k$. We have a lift of groups from (2.1):

$$\begin{array}{ccccc}
 C(\pi) & \longrightarrow & C_{\text{Diff}(\tilde{M})}(\pi) & \longrightarrow & \text{Diff}(M)^0 \leq \ker \varphi \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{Z}^k & \longrightarrow & \mathbb{R}^k & \longrightarrow & T^k.
 \end{array}
 \quad (2.2)$$

Since $\mathbb{Z}^k \leq C(\pi)$, letting $Q = \pi/\mathbb{Z}^k$, there is a central group extension:

$$(2.3) \quad 1 \rightarrow \mathbb{Z}^k \rightarrow \pi \rightarrow Q \rightarrow 1.$$

Now \mathbb{R}^k acts properly and freely on \tilde{M} such that $\tilde{M} = \mathbb{R}^k \times W$ where $W = \tilde{M}/\mathbb{R}^k$ is a simply connected smooth manifold. The group extension (2.3) represents a 2-cocycle f in $H^2(Q; \mathbb{Z}^k)$ in which π is viewed as the product $\mathbb{Z}^k \times Q$ with group law:

$$(n, \alpha)(m, \beta) = (n + m + f(\alpha, \beta), \alpha\beta).$$

Let $\text{Map}(W, \mathbb{R}^k)$ (respectively $\text{Map}(W, T^k)$) be the set of smooth maps of W into \mathbb{R}^k (respectively T^k) endowed with a Q -module structure in which

there is an exact sequence of Q -modules [7]:

$$1 \rightarrow \mathbb{Z}^k \rightarrow \text{Map}(W, \mathbb{R}^k) \xrightarrow{\exp} \text{Map}(W, T^k) \rightarrow 1.$$

As Q acts properly discontinuously on W with compact quotient, it follows from [7, Lemma 8.5] (also [14]):

$$(2.4) \quad H^i(Q; \text{Map}(W, \mathbb{R}^k)) = 0 \quad (i \geq 1)$$

so that the connected homomorphism $\delta : H^1(Q; \text{Map}(W, T^k)) \rightarrow H^2(Q; \mathbb{Z}^k)$ is an isomorphism. From this, there exists a map $\chi : Q \rightarrow \text{Map}(W, \mathbb{R}^k)$ such that $\delta^1 \chi = f$. Then the π -action on \tilde{M} can be described as

$$(2.5) \quad \begin{aligned} (n, \alpha)(x, w) &= (n + x + \chi(\alpha)(\alpha w), \alpha w) \\ (\forall (n, \alpha) \in \pi, \forall (x, w) \in \mathbb{R}^k \times W). \end{aligned}$$

The action of π may depend on the choice of χ' such that $\delta^1 \chi' = f$. However, the vanishing of (2.4) shows

Proposition 2.1. *Such π -actions are equivalent to each other.*

Let (T^k, M) be an injective T^k -action on a closed manifold M which induces a central group extension (2.3) as above.

Definition 2.2. A T^k -action is said to be *injective-splitting* if there exists a finite index normal subgroup Q' of Q such that the induced extension splits;

$$\pi' = \mathbb{Z}^k \times Q'.$$

Here is a key result concerning *injective-splitting torus actions*.

Theorem 2.3. *Suppose that a closed manifold M admits an injective-splitting T^k -action. Then the following hold.*

- (i) $kC_j \leq b_j$. In particular the Halperin-Carlsson conjecture is true.
- (ii) The T^k -action is homologically injective.

Proof. Algebraic part. (2.3) induces a commutative diagram:

$$(2.6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \pi & \longrightarrow & Q \longrightarrow 1 \\ & & \parallel & & \uparrow \iota & & \uparrow \iota' \\ 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \pi' & \longrightarrow & Q' \longrightarrow 1 \end{array}$$

Here Q/Q' is a finite group by Definition 2.2. For the cocycle f representing the upper group extension, it follows $\iota'^*[f] = 0 \in H^2(Q'; \mathbb{Z}^k)$ by the hypothesis. We may assume

$$(2.7) \quad f|_{Q'} = 0.$$

On the other hand, if $\tau : H^2(Q'; \mathbb{Z}^k) \rightarrow H^2(Q; \mathbb{Z}^k)$ is the transfer homomorphism (cf. [3], [2]), then $\tau \circ \iota'^* = |Q : Q'| : H^2(Q; \mathbb{Z}^k) \rightarrow H^2(Q; \mathbb{Z}^k)$ so that

$[f]$ is a torsion in $H^2(Q; \mathbb{Z}^k)$. There exists an integer ℓ such that $\ell \cdot f = \delta^1 \tilde{\lambda}$ for some function $\tilde{\lambda} : Q \rightarrow \mathbb{Z}^k$. Put $\lambda = \frac{\tilde{\lambda}}{\ell} : Q \rightarrow \mathbb{R}^k$. Then

$$(2.8) \quad f = \delta^1 \lambda.$$

(2.7) shows $[\lambda|_{Q'}] \in H^1(Q; \mathbb{R}^k)$. Viewed $\mathbb{R}^k \leq \text{Map}(W, \mathbb{R}^k)$ as constant maps, $[\lambda|_{Q'}] \in H^1(Q; \text{Map}(W, \mathbb{R}^k)) = 0$ by (2.4). So there is an element $h \in \text{Map}(W, \mathbb{R}^k)$ such that $\lambda|_{Q'} = \delta^0 h$. The equality $\lambda(\alpha') = \delta^0 h(\alpha')(w)$ ($\forall \alpha' \in Q', \forall w \in W$) implies

$$(2.9) \quad h(w) = h(\alpha' w) + \lambda(\alpha').$$

Geometric part. Noting Proposition 2.1, the π -action (2.5) on \tilde{M} is equivalent with

$$(2.10) \quad (n, \alpha)(x, w) = (n + x + \lambda(\alpha), \alpha w) \quad (\forall (x, w) \in \mathbb{R}^k \times W).$$

Recall that π has the splitting subgroup $\pi' = \mathbb{Z}^k \times Q'$. Obviously we have the product action of $\mathbb{Z}^k \times Q'$ on $\mathbb{R}^k \times W$ such that $\mathbb{R}^k \times W / \mathbb{Z}^k \times Q' = T^k \times W / Q'$. Define a diffeomorphism $\tilde{G} : \mathbb{R}^k \times W \rightarrow \mathbb{R}^k \times W$ to be $\tilde{G}(x, w) = (x + h(w), w)$. Using (2.9), it is easy to check that $\tilde{G} : (\pi', \mathbb{R}^k \times W) \rightarrow (\mathbb{Z}^k \times Q', \mathbb{R}^k \times W)$ is an equivariant diffeomorphism with respect to the action (2.10) and the product action. Putting $\mathbb{R}^k \times W / \pi' = T^k \times W / Q'$ as a quotient space, \tilde{G} induces a diffeomorphism $G : T^k \times W \rightarrow T^k \times W / Q'$. Let $q : T^k \times W \rightarrow T^k \times W / Q'$ be the covering map ($q(t, w) = [t, w]$). Then

$$(2.11) \quad G \circ q(t, w) = G([t, w]) = (t \exp 2\pi i h(w), [w]).$$

Noting (2.10), π induces an action of Q on $\tilde{M} / \mathbb{Z}^k = T^k \times W$ such that

$$(2.12) \quad \alpha(t, w) = (t \exp 2\pi i \lambda(\alpha), \alpha w) \quad (\forall \alpha \in Q).$$

$F = Q / Q'$ has an induced action on $T^k \times W / Q'$ by $\hat{\alpha}[t, w] = [t \exp 2\pi i \lambda(\alpha), \alpha w]$ ($\forall \hat{\alpha} \in F$) which gives rise to a covering map:

$$(2.13) \quad F \rightarrow T^k \times W / Q' \xrightarrow{\nu} T^k \times W / Q = M.$$

For any $\alpha \in Q$, consider the commutative diagram:

$$(2.14) \quad \begin{array}{ccc} H_j(T^k \times W) & \xrightarrow{\alpha_*} & H_j(T^k \times W) \\ \downarrow q_* & & \downarrow q_* \\ H_j(T^k \times_{Q'} W) & \xrightarrow{\hat{\alpha}_*} & H_j(T^k \times_{Q'} W) \end{array}$$

in which $H_j(T^k) \otimes H_0(W) \leq H_j(T^k \times W)$. By the formula (2.12), the Q -action on the T^k -summand is a translation by $\exp 2\pi i \lambda(\alpha) \in T^k$ so the homology action α_* on $H_j(T^k) \otimes H_0(W)$ is trivial. If $H_j(T^k \times W / Q')^F$ denotes

the subgroup left fixed under the homology action for every element $\hat{\alpha} \in F$, it follows

$$(2.15) \quad q_*(H_j(T^k) \otimes H_0(W)) \leq H_j(T^k \times_{Q'} W)^F.$$

Using the transfer homomorphism ([2, 2.4 Theorem, III], [3]), ν of (2.13) induces an isomorphism: $\nu_* : H_j(T^k \times_{Q'} W; \mathbb{Q})^F \longrightarrow H_j(M; \mathbb{Q})$. In particular,

$\nu_* : q_*(H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q})) \rightarrow H_j(M; \mathbb{Q})$ is injective.

On the other hand, let $q' : W \rightarrow W/Q'$ be the projection $q'(w) = [w]$. Define a homotopy $\Psi_\theta : T^k \times W \rightarrow T^k \times W/Q'$ ($\theta \in [0, 1]$) to be

$$\Psi_\theta(t, w) = (t \exp 2\pi i(\theta \cdot h(w)), [w]).$$

Then $\Psi_0 = \text{id} \times q' \simeq G \circ q$ from (2.11). As $G_* \circ q_* = \text{id} \times q'_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \rightarrow H_j(T^k; \mathbb{Q}) \otimes H_0(W/Q'; \mathbb{Q})$ is obviously isomorphic, it implies that $q_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \rightarrow H_j(T^k \times_{Q'} W; \mathbb{Q})$ is injective. If $p = \nu \circ q : T^k \times W \rightarrow M$ is the projection, then $p_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \rightarrow H_j(M; \mathbb{Q})$ is injective.

As $p : T^k \times W \rightarrow M$ is T^k -equivariant, letting $p(1, w) = x \in M$, it follows $p(t, w) = tx = \text{ev}(t)$ ($\forall t \in T^k$). Define an embedding $\tilde{\text{ev}} : T^k \rightarrow T^k \times W$ to be $\tilde{\text{ev}}(t) = (t, w)$. Obviously $\tilde{\text{ev}}_* : H_j(T^k; \mathbb{Q}) \rightarrow H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q})$ is an isomorphism. Since $p \circ \tilde{\text{ev}}(t) = \text{ev}(t)$, chasing a commutative diagram:

$$(2.16) \quad \begin{array}{ccc} H_j(T^k; \mathbb{Q}) & \xrightarrow{\tilde{\text{ev}}_*} & H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \\ \text{ev}_* \searrow & & \swarrow p_* \\ & H_j(M; \mathbb{Q}) & \end{array}$$

$\text{ev}_* : H_j(T^k; \mathbb{Q}) \rightarrow H_j(M; \mathbb{Q})$ is injective. As $H_j(T^k; \mathbb{Z})$ has no torsion, $\text{ev}_* : H_j(T^k; \mathbb{Z}) \rightarrow H_j(M; \mathbb{Z})$ turns out to be injective. This proves (i) ${}_k C_j \leq b_j$. In particular, $\text{ev}_* : \mathbb{Z}^k \rightarrow H_1(M; \mathbb{Z})$ is injective for $j = 1$, i.e. the T^k -action is homologically injective by the definition. This shows (ii). \square

Any homologically injective action is obviously injective. Theorem A is obtained from the following corollary.

Corollary 2.4. *If T^k is a homologically injective action on a closed manifold M , then ${}_k C_j \leq b_j$. Thus the Halperin-Carlsson conjecture is true.*

Proof. The proof is essentially the same as [8, 2.2. Lemma]. Let $1 \rightarrow \mathbb{Z}^k \rightarrow \pi \rightarrow Q \rightarrow 1$ be the central group extension. As $\text{ev}_* : H_1(T^k; \mathbb{Z}) = \mathbb{Z}^k \rightarrow H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell \oplus F$ is injective, $\text{ev}_*(\mathbb{Z}^k) \leq \mathbb{Z}^k$ such that $\text{ev}_*(\mathbb{Z}^k) \oplus \mathbb{Z}^{\ell-k} \leq \mathbb{Z}^\ell$. If $q : \pi \rightarrow H_1(M; \mathbb{Z})$ is a canonical projection, then $\pi' = q^{-1}(\text{ev}_*(\mathbb{Z}^k) \oplus \mathbb{Z}^{\ell-k} \oplus F)$ is a finite index normal splitting subgroup of π . \square

Remark 2.5. *As a consequence of this corollary and (ii) of Theorem 2.3, injective-splitting action is equivalent with homologically injective action.*

Let (M, g) be a $2n$ -dimensional Kähler manifold with Kähler form Ω .

Corollary 2.6 ([5]). *Every holomorphic action of a complex torus $T_{\mathbb{C}}^k$ on a compact Kähler manifold (M, Ω) is homologically injective. Thus the Halperin-Carlsson conjecture holds.*

Proof. Averaging the Kähler metric by $T_{\mathbb{C}}^k$, we may assume that $T_{\mathbb{C}}^k$ acts as Kähler isometries on M . $T_{\mathbb{C}}^k$ induces the Killing vector fields $\xi_i, J\xi_i$ on M ($i = 1, \dots, k$). Note that each ξ_i is a non-vanishing vector field by the maximum principle. In fact, if G is the connected component of the stabilizer of $T_{\mathbb{C}}^k$ at $x \in M$, then the action induces a holomorphic representation $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$. As G is compact, ρ is trivial so that $G = \{1\}$.

Put $\theta_i = \iota_{\xi_i} \Omega$ and $\theta_{i+k} = \iota_{J\xi_i} \Omega$ ($i = 1, \dots, k$). By the Cartan formula, $d\Omega = 0$ implies $d\theta_i = 0$ ($i = 1, \dots, 2k$). We have $2k$ -number of 1-cocycles $[\theta_i] \in H^1(M; \mathbb{R})$. As $\mathrm{ev}(t) = t \cdot x \in M$, it follows $\mathrm{ev}_*((\xi_i)_1) = (\xi_i)_x$. Since $\mathrm{ev}^*\theta_i((\xi_{k+i})_1) = \Omega((\xi_i)_x, (J\xi_i)_x) = g(\xi_i, \xi_i) > 0$ on $T_{\mathbb{C}}^k$, $\mathrm{ev}^* : H^1(M; \mathbb{R}) \rightarrow H^1(T_{\mathbb{C}}^k; \mathbb{R})$ is surjective. So $\mathrm{ev}_* : H_1(T_{\mathbb{C}}^k; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$ is injective. \square

For example, any holomorphic action of $T_{\mathbb{C}}^k$ on a compact complex euclidean space form is homologically injective. Recall that any effective T^s -action on a closed aspherical manifold is injective. We prove Corollary B.

Theorem 2.7. *Any effective T^s -action on a compact euclidean space form M is homologically injective. Thus the Halperin-Carlsson conjecture is true.*

Proof. Given a T^s -action for some $s \geq 1$, there is a central group extension: $1 \rightarrow \mathbb{Z}^s \rightarrow \pi = \pi_1(M) \rightarrow Q \rightarrow 1$. Since π has a unique maximal normal finite index abelian subgroup \mathbb{Z}^n (cf. [20]), it follows $\mathbb{Z}^s \leq \mathbb{Z}^n$. Q has a finite index subgroup $Q' = \mathbb{Z}^n / \mathbb{Z}^s \cong G \times \mathbb{Z}^{n-s}$ where G is a finite abelian group. The inclusion $\iota : \mathbb{Z}^{n-s} \rightarrow Q'$ induces a group (extension) A of finite index in \mathbb{Z}^n . As A is isomorphic to $\mathbb{Z}^s \times \mathbb{Z}^{n-s}$, A is a finite index normal splitting subgroup of π . The T^s -action is injective-splitting so apply Theorem 2.3. \square

3. CALABI CONSTRUCTION AND TORUS ACTIONS

In [8, § 7], Conner and Raymond have stated that the Calabi's theorem [4] shows the existence of a T^k -action with $k = \mathrm{rank} H_1(M; \mathbb{Z}) > 0$. We agree the existence of such actions in view of the Calabi construction. However when we look at a proof of the Calabi's theorem ([20, p.125]), it is not easy to find such T^k -actions. In fact, let $\nu : \pi \rightarrow \mathbb{Z}^k$ be the projection onto the direct summand \mathbb{Z}^k of $H_1(M; \mathbb{Z})$. Then there is a group extension $1 \rightarrow \Gamma \rightarrow \pi \rightarrow \mathbb{Z}^k \rightarrow 1$ in which Γ is the fundamental group of a euclidean space form $M^{n-k} = \mathbb{R}^{n-k} / \Gamma$. In general an element $\gamma \in \pi$ has the form

$$\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \right) \quad (a \in \mathbb{R}^{n-k}, b \in \mathbb{R}^k).$$

The holonomy group $L(\pi) = \left\{ \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \right\}$ does not necessarily leave the subspace $0 \times \mathbb{R}^k$ invariant. (In particular, $\mathbb{Z}^n \cap (0 \times \mathbb{R}^k)$ is not necessarily

uniform in $0 \times \mathbb{R}^k$.) So we have to find another decomposition to get a T^k -action on M .

Lemma 3.1. *Let π be a Bieberbach group such that $\text{rank } \pi/[\pi, \pi] = k > 0$. Then there exists a faithful representation $\rho : \pi \rightarrow \text{E}(n)$ such that the euclidean space form $\mathbb{R}^n/\rho(\pi)$ admits an effective T^k -action.*

Proof. By the hypothesis, there is a group extension $1 \rightarrow \Gamma \rightarrow \pi \xrightarrow{\nu} \mathbb{Z}^k \rightarrow 1$. Since π is a Bieberbach group, it admits a maximal normal finite index abelian subgroup \mathbb{Z}^n . Put $\nu(\mathbb{Z}^n) = A$. Consider the commutative diagram of the group extensions:

$$(3.1) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \xrightarrow{\iota} & \pi & \xrightarrow{\nu} & \mathbb{Z}^k & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Gamma \cap \mathbb{Z}^n & \xrightarrow{\iota} & \mathbb{Z}^n & \xrightarrow{\nu} & A & \longrightarrow & 1. \end{array}$$

Since $\pi/\mathbb{Z}^n \xrightarrow{\hat{\nu}} \mathbb{Z}^k/A$ is surjective, A is a free abelian subgroup of rank k . By the embedding $\hat{\iota} : \Gamma/\Gamma \cap \mathbb{Z}^n \leq \pi/\mathbb{Z}^n$, $\Gamma \cap \mathbb{Z}^n$ is a finite index subgroup of Γ . It follows easily that $\Gamma \cap \mathbb{Z}^n$ is a maximal normal abelian subgroup of Γ . We may put $\Gamma \cap \mathbb{Z}^n = \mathbb{Z}^{n-k}$ so that $\mathbb{Z}^n = \mathbb{Z}^{n-k} \times A$. Putting $Q = \pi/\mathbb{Z}^{n-k}$ and $F = \Gamma/\mathbb{Z}^{n-k}$ is a finite group, we have the group extensions:

$$(3.2) \quad 1 \longrightarrow \mathbb{Z}^{n-k} \xrightarrow{i} \pi \xrightarrow{\mu} Q \longrightarrow 1,$$

where

$$(3.3) \quad 1 \longrightarrow F \xrightarrow{\hat{\iota}} Q \xrightarrow{\hat{\nu}} \mathbb{Z}^k \longrightarrow 1$$

is also a group extension. Consider the commutative diagram of group extensions:

$$(3.4) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^{n-k} & \xrightarrow{\iota} & \pi & \xrightarrow{\mu} & Q & \longrightarrow & 1 \\ & & \parallel & & \uparrow & & \uparrow \iota' & & \\ 1 & \longrightarrow & \mathbb{Z}^{n-k} & \xrightarrow{\iota} & \mathbb{Z}^n & \xrightarrow{\mu} & B & \longrightarrow & 1 \end{array}$$

where we put $B = \mu(\mathbb{Z}^n)$. As $\hat{\nu}(B) = \nu(\mathbb{Z}^n) = A$ from (3.2), it follows $\mathbb{Z}^n = \mathbb{Z}^{n-k} \times B$. Thus \mathbb{Z}^n is a splitting subgroup of π . Since (3.2) is not necessarily central, let $\phi : Q \rightarrow \text{Aut}(\mathbb{Z}^{n-k})$ be the conjugation homomorphism. If $[f] \in H_\phi^2(Q; \mathbb{Z}^k)$ is the representative cocycle of (3.2), then $\iota'^*[f] = 0$ in $H^2(B; \mathbb{Z}^{n-k})$. Then $[f]$ is a torsion because $\tau \circ \iota'^* = |Q/B| : H_\phi^2(Q; \mathbb{Z}^k) \rightarrow H_\phi^2(Q; \mathbb{Z}^k)$ still holds for the transfer homomorphism $\tau : H^2(B; \mathbb{Z}^{n-k}) \rightarrow H_\phi^2(Q; \mathbb{Z}^{n-k})$. (Compare [3].) Similarly as in the proof of Theorem 2.3 there is a function $\lambda : Q \rightarrow \mathbb{R}^{n-k}$ such that $f = \delta^1 \lambda$. Note from (2.8) that

$$(3.5) \quad \ell \cdot \lambda(Q) \leq \mathbb{Z}^{n-k}.$$

Let \mathbb{Z}^k act on \mathbb{R}^k by translations and by (3.3) $\hat{\nu} : Q \rightarrow \mathbb{Z}^k \leq \text{E}(k)$ defines a properly discontinuous action of Q on \mathbb{R}^k ;

$$\alpha(w) = \hat{\nu}(\alpha) + w \quad (\forall \alpha \in Q, \forall w \in \mathbb{R}^k).$$

As in (2.5) we have a properly discontinuous action of π on $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$:

$$\begin{aligned} (n, \alpha) \begin{bmatrix} x \\ w \end{bmatrix} &= \begin{bmatrix} n + \bar{\phi}(\alpha)(x) + \lambda(\alpha) \\ \hat{\nu}(\alpha) + w \end{bmatrix} \\ &= \left(\begin{bmatrix} n + \lambda(\alpha) \\ \hat{\nu}(\alpha) \end{bmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & \\ & I_k \end{pmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix}. \end{aligned}$$

Since $\phi|_B = \text{id}$ from (3.4), the image $\phi(Q)$ is finite in $\text{Aut}(\mathbb{Z}^{n-k})$ which implies $\phi(Q) \leq O(n-k)$ up to conjugate. As π is torsionfree and acts properly discontinuously, we obtain a faithful homomorphism $\rho : \pi \rightarrow \rho(\pi) \leq E(n)$ defined by

$$(3.6) \quad \rho(n, \alpha) = \left(\begin{bmatrix} n + \lambda(\alpha) \\ \hat{\nu}(\alpha) \end{bmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & \\ & I_k \end{pmatrix} \right).$$

Therefore we have a compact euclidean space form $\mathbb{R}^n / \rho(\pi)$.

We prove that $\mathbb{R}^n / \rho(\pi)$ admits a T^k -action. Noting (3.5), we define a subgroup of \mathbb{Z}^n by

$$(3.7) \quad \tilde{B} = \{(-\ell \cdot \lambda(\beta), \ell \cdot \beta) \in \mathbb{Z}^n \mid \beta \in B\}.$$

It is isomorphic to $B \cong \mathbb{Z}^k$. As $\phi|_B = \text{id}$,

$$(3.8) \quad \rho(-\ell \cdot \lambda(\beta), \ell \cdot \beta) = \left(\begin{bmatrix} 0 \\ \ell \cdot \hat{\nu}(\beta) \end{bmatrix}, I_n \right) \in 0 \times \mathbb{R}^k.$$

Thus $\rho(\tilde{B})$ is a translation subgroup with rank k :

$$(3.9) \quad \rho(\tilde{B}) \leq (0 \times \mathbb{R}^k) \cap \rho(\pi).$$

Since $(0 \times \mathbb{R}^k) / \rho(\tilde{B})$ is compact, so is $(0 \times \mathbb{R}^k) / (0 \times \mathbb{R}^k) \cap \rho(\pi)$. We may put $(0 \times \mathbb{R}^k) / (0 \times \mathbb{R}^k) \cap \rho(\pi) = T^k$. Moreover, from (3.6) a calculation shows that

$$(3.10) \quad \rho(n, \alpha) \cdot \left(\begin{bmatrix} 0 \\ y \end{bmatrix}, I_n \right) = \left(\begin{bmatrix} 0 \\ y \end{bmatrix}, I_n \right) \cdot \rho(n, \alpha),$$

i.e. each $y \in \mathbb{R}^k$ centralizes $\rho(\pi)$;

$$(3.11) \quad 0 \times \mathbb{R}^k \leq C_{E(n)}(\rho(\pi)).$$

Let $\text{Isom}(\mathbb{R}^k / \rho(\pi))^0$ denote the identity component of euclidean isometries of $\mathbb{R}^k / \rho(\pi)$. From (3.11) we have the following covering groups (cf. (2.2)):

$$\begin{array}{ccccccc} 1 & \longrightarrow & C(\rho(\pi)) & \longrightarrow & C_{E(n)}(\rho(\pi)) & \longrightarrow & \text{Isom}(\mathbb{R}^k / \rho(\pi))^0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & (0 \times \mathbb{R}^k) \cap \rho(\pi) & \longrightarrow & (0 \times \mathbb{R}^k) & \longrightarrow & T^k \end{array}$$

Hence $\mathbb{R}^k / \rho(\pi)$ admits a T^k -action. □

When the center $C(\pi)$ of $\pi = \pi_1(M)$ for a closed manifold M is finitely generated, $\text{rank } C(\pi)$ denotes the rank of a free abelian subgroup. Recall from [17] that an effective T^s -action on M is said to be *maximal* if $s = \text{rank } C(\pi)$. If M is a closed aspherical manifold admitting an effective T^s -action, then $s \leq \text{rank } C(\pi)$ (cf. [7]). Let M be a euclidean space form $M = \mathbb{R}^n/\pi$. If $C(\pi)$ has rank k , then it is easy to see that M admits a T^k -action. In particular, $\text{rank } C(\pi) \leq \text{rank } H_1(M; \mathbb{Z})$ by Theorem 2.3.

The Bieberbach theorem implies that M is affinely diffeomorphic to $\mathbb{R}^n/\rho(\pi)$. Combined Lemma 3.1 with Theorem 2.7, we obtain

Theorem 3.2. *Let M be a compact n -dimensional euclidean space form with $\text{rank } H_1(M; \mathbb{Z}) = k > 0$. Then M admits a homologically injective T^k -action. In particular, $\text{rank } C(\pi) = \text{rank } H_1(M; \mathbb{Z})$.*

Acknowledgement. We thank Professor M. Masuda who called our attention to the Halperin-Carlsson conjecture.

REFERENCES

1. O. Baues, *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups*, Topology, **43** (2004), no. 4, 903–924.
2. G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.
3. K. Brown, *Cohomology of groups*, GTM, Springer-Verlag, 1982.
4. E. Calabi, *Closed locally euclidean 4-dimensional manifolds*, Bull. of AMS, 63 (1957) p.135.
5. J. Carrell, *Holomorphically injective complex toral actions*, Proc. Second conference on Compact transformation groups, Part II, Lecture notes in Math., 299, Springer, New York 1972 205–236.
6. S. Choi, M. Masuda and D. Y. Suh, *Topological classification of generalized Bott towers*, preprint.
7. P.E. Conner and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, Topology of Manifolds, Proceedings Inst. Univ. of Georgia, Athens, 1969, Markham (1970), 227–264.
8. P.E. Conner and F. Raymond, *Injective operations of the toral groups*, Topology 10, (1971) 283–296.
9. M. Grossberg and Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Math. J. 76 (1994) 23–58.
10. S. Halperin, *Rational homotopy and torus actions*, Aspects of Topology, London Math. Soc. Lecture Note Ser. 93 (1985), 293–306.
11. Y. Kamishima, K.B. Lee and F. Raymond, *The Seifert construction and its applications to infranil manifolds*, Quart. J. Math., Oxford (2), **34** (1983), 433–452.
12. Y. Kamishima and Admi Nazra, *Seifert fibred structure and rigidity on real Bott towers*, Contemp. Math., vol. 501, (2009), 103–122.
13. J.B. Lee and M. Masuda, *Topology of iterated S^1 -bundles*, arXiv:1108.0293 (math.AT.) 2011.
14. K.B. Lee and F. Raymond, *Seifert fiberings*, Mathematical Surveys and Monographs, vol. 166, 2010.
15. M. Nakayama, *On the S^1 -fibred nil-Bott Tower*, arXiv:1110.1164 (math.AT.) 2011.
16. K.B. Lee and F. Raymond, *Topological, affine and isometric actions on flat Riemannian manifolds*, J. Differential Geom. **16** (1981), 255–269.

17. K.B. Lee and F. Raymond, *Maximal torus actions on solvmanifolds and double coset spaces*, *Int. J. Math.* **92** (1991), 67-76.
18. M. Masuda and T. Panov, *Semi-free circle actions, Bott towers, and quasitoric manifolds*, *Sb. Math.* **199** (2008), no. 7-8, 1201-1223
19. V. Puppe, *Multiplicative aspects of the Halperin-Carlsson conjecture*, *Georgian Math. J.* **16** (2009), no. 2, 369-379.
20. J. Wolf, *Spaces of constant curvature*, McGraw-Hill, Inc., 1967.

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, MINAMI-OHSAWA
1-1, HACHIOJI, TOKYO 192-0397, JAPAN

E-mail address: kami@tmu.ac.jp

DEPARTMENT OF MATHEMATICS AND INFORMATION OF SCIENCES, TOKYO METRO-
POLITAN UNIVERSITY, MINAMI-OHSAWA 1-1, HACHIOJI, TOKYO 192-0397, JAPAN

E-mail address: nakayama-mayumi@ed.tmu.ac.jp